

Noise and dynamic transitions

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Abstract

A parabolic stochastic PDE is studied analytically and numerically, when a bifurcation parameter is slowly increased through its critical value. The aim is to understand the effect of noise on delayed bifurcations in systems with spatial degrees of freedom. Realisations of the nonautonomous stochastic PDE remain near the unstable configuration for a long time after the bifurcation parameter passes through its critical value, then jump to a new configuration. The effect of the non-linearity is to freeze in the spatial structure formed from the noise near the critical value.

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1. Introduction

Many physical systems undergo a transition from a spatially uniform state to one of lower symmetry. Such systems are commonly modelled by a simple differential equation which has a bifurcation parameter with a critical value at which there is an exchange of stability between steady states [1]. The model can be improved by adding noise [2], which can be understood as allowing for small rapidly varying effects neglected in the formulation of the model. Noise is also necessary to provide the initial symmetry-breaking which permits the system to choose one of the available lower-symmetry states.

If the bifurcation parameter is not a function of time, the effect of noise is to make a bifurcation ill-defined in an $\mathcal{O}(\epsilon)$ region near the critical value, where ϵ is the size of the noise [3]. A more dramatic effect is found if the bifurcation parameter is a function of time, corresponding to an experimental situation in which the critical value is not known in advance, so the parameter is slowly changed until a qualitative change is seen in the behaviour of the system [4].

In the case of the pitchfork bifurcation, described by the normal form

$$\dot{y} = gy - y^3, \quad (1)$$

it is normally said that, if $g > 0$, y will be found either at $y = \sqrt{g}$ or $y = -\sqrt{g}$. However, on solving the stochastic differential equation

$$dy = (gy - y^3)dt + \epsilon dw \quad (2)$$

with $g = \mu t$, starting with $y = 0$ at $g = g_0 < 0$, it is found that y remains near $y = 0$ until well after $g = 0$ (Figure 1). (The Wiener process – standard Brownian motion – is denoted by lower case w .) If μ is small, a typical trajectory consists of a long sojourn near $y = 0$, a sudden jump away from $y = 0$ and then relaxation towards $y^2 = g$. The value of g at which the jump occurs is a random variable whose probability distribution can be found by solving the linearised version of (2) [5]; the mean value is approximately $\sqrt{2\mu|\ln \epsilon|}$ and the width of the probability distribution is proportional to μ .

The phenomenon of delayed bifurcation just described and its sensitivity to noise has been studied theoretically and experimentally in the case of non-autonomous stochastic ordinary differential equations [6-9]; the corresponding phenomenon, delayed transition, for PDEs is the subject of this article. A dynamic transition is described by the following non-autonomous parabolic stochastic partial differential equation:

$$dY = (g(t)Y - Y^3 + \Delta Y)dt + \epsilon dW, \quad (3)$$

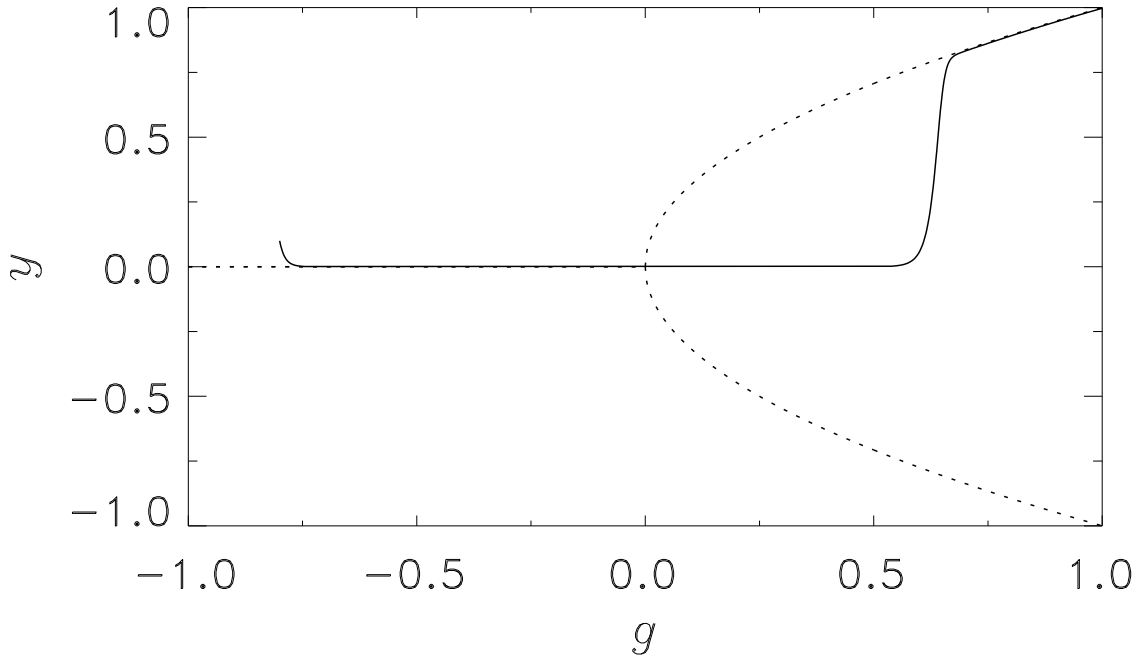


Figure 1. *Dynamic pitchfork bifurcation with noise.* The solid line is one trajectory of the ordinary SDE (2), with $\epsilon = 10^{-10}$, $\mu = 0.01$, and initial condition $y = 0.1$ at $g = -0.8$. Also shown, as dotted lines, are the loci of stable fixed points of the corresponding autonomous system (1). The trajectory lingers near $y = 0$ for a long time after g passes through 0; the value of g at which the jump towards $y = \pm\sqrt{g}$ occurs is a random variable with mean approximately equal to $\sqrt{2\mu|\ln \epsilon|}$.

where $g = \mu t$ is slowly increased through 0, W is the Brownian sheet, $\Delta = \sum_{i=1}^m \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in \mathbf{R}^m (the derivatives exist only in the sense of generalised functions) and μ and ϵ are constants such that

$$\epsilon \ll \mu \ll 1. \quad (4)$$

In section 2 this initial value problem is solved using periodic boundary conditions in one space dimension, in which case Y is a stochastic process with values in the space of real-valued continuous functions on $[0, L]$, and has continuous mean $\langle Y_t(x) \rangle$ and correlation function $\langle Y_t(x)Y_t(x') \rangle$ [10, 11]. In section 3 the corresponding equation in space dimension, m , greater than one is discussed.

The present work was motivated by the behaviour of some nonlinear systems of ordinary and partial differential equations whose dynamics are controlled by noise [12, 13]. The controlling influence of noise arises because trajectories spend long times near a slow invariant manifold, and the dynamics near this manifold are similar to those of a dynamic bifurcation [5]. The most important parameter in the

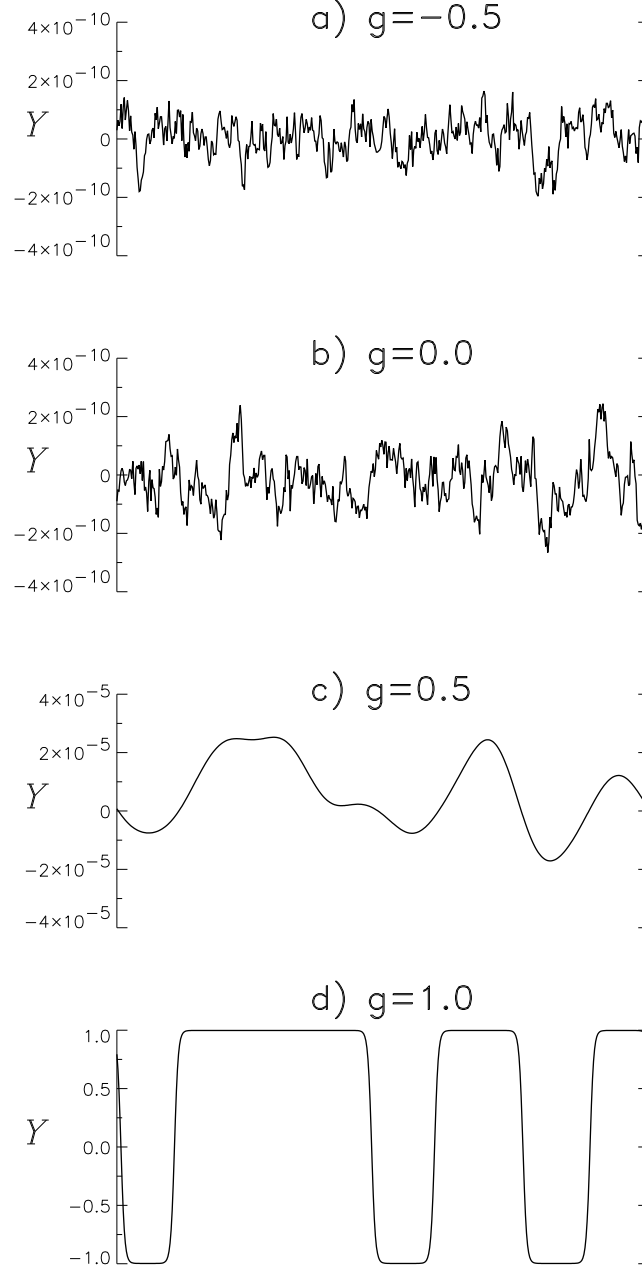


Figure 2. *Dynamic transition in one space dimension.* One numerically-generated realisation of the stochastic PDE (3) $Y_t(x)$, $x \in [0, L]$, is displayed at four times as the bifurcation parameter, $g = \mu t$, is slowly increased, passing through 0 at $t = 0$. (Note the different vertical scales.) It remains close to the zero configuration until well after $g = 0$. Nonlinear terms become important when $g \simeq \sqrt{2\mu|\ln \epsilon|}$; their effect is to freeze in the spatial structure formed, near $g = 0$, from the noise. The data shown here were generated using a finite difference method with 500 points. ($L = 300$, $\mu = 0.01$, $\epsilon = 10^{-10}$.)

description of these systems is $\mu|\ln \epsilon|$, where $\frac{1}{\mu}$ is the timescale for the dynamics near the slow manifold and ϵ is the magnitude of the noise.

2. Model dynamic transition in one space dimension

Four configurations from a numerical realisation of (3) with $m = 1$ are shown in Figure 2. According to static stability analysis the zero configuration is unstable for $g > 0$, but Y typically remains much closer to 0 than to $Y^2 = g$ until well after $g = 0$ when g is time-dependent. The first part of a dynamic transition, the loss of stability of the zero configuration, is thus controlled by the noise and the sweep rate, not by the nonlinearity.

If Y obeys (3), the linearised version of the stochastic differential equation for its k th Fourier mode, $u_t^{(k)} = \frac{1}{\sqrt{L}} \int_0^L Y_t(x') e^{ikx2\pi/L} dx'$, is

$$du^{(k)} = (g(t) - (\frac{2\pi}{L})^2 k^2) dt + \epsilon dw^{(k)}, \quad (5)$$

with each $w^{(k)}$ an independent Wiener process. Thus, for as long as nonlinear terms are unimportant, each Fourier mode evolves independently, satisfying an equation of the dynamic bifurcation type.

As the in ODE case, the new configuration (Figure 2(d)) appears abruptly at a value of g which is a random variable with mean approximately $\sqrt{2\mu|\ln\epsilon|}$. The effect of the cubic nonlinearity is to arrest the exponential growth of $\langle Y_t(x)Y_t(x) \rangle$ when it becomes $\mathcal{O}(1)$ and to freeze in the spatial structure formed during the slow sweep. Establishing when this happens makes it possible to answer another question which is not considered in the static version: what is the typical size of the spatial domains formed in the transition?

To examine the dynamics of the loss of stability of the zero configuration, consider the solution of the linearised version of (3),

$$dY = (g(t)Y + \Delta Y)dt + \epsilon dW, \quad (6)$$

with initial data $Y_t(x) = f(x)$ on $x \in [0, L]$ at $t = t_0 < 0$ when $g = g_0$:

$$Y(x, t) = \int_0^L G(t, t_0, x, x') f(x') dx' + \epsilon \int_{t_0}^t \int_{[0, L]} G(t, t_0, x, x') dW_t(x'). \quad (7)$$

The first integral is an ordinary integral; the second a space-time Itô integral. The fundamental solution $G(t, t_0, x, x')$ is given by

$$G(t, t', x, x') = \frac{1}{\sqrt{4\pi(t-t')}} e^{\frac{1}{2}\mu(t^2-t'^2)} \sum_{j=-\infty}^{\infty} e^{-\frac{(x-x'-jL)^2}{4(t-t')}}. \quad (8)$$

The solution (7) prescribes the mean and correlation function of Y as a function of time. The mean value of Y is the first term, an ordinary integral which, for large $t - t_0$, is given by

$$\langle Y_t(x) \rangle = e^{\frac{1}{2}\mu(t^2-t_0^2)} \bar{y} \quad (9)$$

where \bar{y} is a smoothed version of the initial data $f(x)$. In the rest of this article, it is assumed that $\bar{y} \leq \mathcal{O}(1)$ and $2\mu|\ln \epsilon| < g_0^2$. This ensures that $\langle Y_t \rangle^2 \ll \langle Y_t^2 \rangle$, ie Y is taken as a mean-zero Gaussian process. It is also assumed that $|t_0| > \frac{1}{\sqrt{\mu}}$, so that quasi-static equilibrium is reached before $g = 0$, and $L \gg \sqrt{t - t_0}$ so that, at any one x , only one term in (8) is important.

The correlation function is a mean square quantity which can be evaluated using the ‘delta function’ property of the Brownian sheet as an integrator:

$$\begin{aligned} \langle Y_t(x)Y_t(x') \rangle &= \epsilon^2 \int_0^L \int_{t_0}^t G^2(t, s, x, x') ds dx' \\ &= \epsilon^2 \int_{t_0}^t ds \frac{e^{\mu(t^2 - s^2)}}{\sqrt{8\pi(t - s)}} e^{-\frac{(x - x')^2}{8(t - s)}}. \end{aligned} \quad (10)$$

For times t such that $-t > 1/\sqrt{\mu}$ the nonautonomous correlation function (10) differs only by $\mathcal{O}(\mu)$ from the result obtained for fixed $g < 0$ if $t_0 \rightarrow \infty$ [14]:

$$\langle Y_t(x)Y_t(x') \rangle = \frac{\epsilon^2}{4\sqrt{|g|}} e^{-|x - x'|\sqrt{|g|}}. \quad (11)$$

In the non-autonomous case, the correlation function remains finite as t passes through 0, and for large positive times ($t > 1/\sqrt{\mu}$), it is well approximated by:

$$\langle Y_t(x)Y_t(x') \rangle \simeq \epsilon^2 \frac{e^{\mu t^2}}{\sqrt{8\mu t}} e^{-(x - x')^2/8t}. \quad (12)$$

Note the Gaussian form of the spatial correlation and the existence of a characteristic length $\sqrt{8t}$ at time t . This formula is valid until $g \simeq \sqrt{2\mu|\ln \epsilon|}$, when the cubic nonlinearity becomes important. The correlation length at this time becomes the characteristic size of the spatial domains formed (Figure 2(d)).

3. Higher space dimensions

Proceeding as in section 2, the expression obtained for the correlation function at time t is

$$\langle Y_t(x)Y_t(x') \rangle = \int_{t_0}^t ds \frac{e^{\mu(t^2 - s^2)}}{(8\pi(t - s))^{\frac{m}{2}}} e^{-\frac{\|x - x'\|^2}{8(t - s)}}, \quad (13)$$

where $\|x - x'\|$ is the Euclidean norm on \mathbf{R}^m . The major difference from the case $m = 1$ is that the correlation function does not approach a finite limit as $\|x - x'\| \rightarrow 0$. This is a general feature of second order stochastic PDEs in more than one space dimension [15, 16]. On a finite grid, equation (6), with $g < 0$ fixed, is of interest because it generates a stochastic process with non-zero correlation length in both space and time [17]. The singularity in the correlation function at

$x = x'$ manifests itself in the normalisation, which increases without a finite limit as the grid spacing is decreased. In the nonautonomous version the same divergence is seen (logarithmic when $m = 2$), but for $g > 0$ the singularity is only noticeable on very small length scales. The nature of these divergences, in both discrete and continuous versions, is conveniently studied in Fourier space, where the linearised equation separates into uncoupled ordinary SDEs, each of which can be solved exactly.

The finite difference method for a parabolic SPDE consists of replacing the infinite dimensional system (3) by N^m ordinary SDEs on a grid of equally-spaced points in $[0, L]^m$. For the SPDE (3), the SDE at each point is

$$dY(x) = (gY(x) - Y^3(x))dt + \rho\tilde{\Delta}Y(x)dt + \rho^{\frac{m}{4}}\epsilon dw \quad (14)$$

where the discrete Laplacian $\tilde{\Delta}$ is defined by

$$\tilde{\Delta}Y(x) = \sum_{x'} Y(x') - 2mY(x) \quad (15)$$

and the sum is over the $2m$ nearest neighbours of x . Each of the N^m ordinary SDEs has an associated Wiener process, independent of all the others, and so can be numerically solved in the normal way [18]. Note the scaling of the magnitude of the noise added at each grid point with ρ . The smallest length scale resolved is roughly the distance between nearest neighbours, $\frac{1}{\sqrt{\rho}}$. As in numerical solution of deterministic parabolic PDEs, there is a maximum timestep, proportional to $\frac{1}{\rho}$, if the finite difference method is to be numerically stable [19].

Solved on a finite grid in this way, the behaviour of realisations of the non-autonomous version SPDE (3), with $m > 1$ and $g = \mu t$ as in section 2, is qualitatively similar to that in one space dimension [20]. That is, a Gaussian form of the spatial correlation function emerges from the slow sweep through $g = 0$,

$$\langle Y_t(x)Y_t(x') \rangle \simeq \epsilon^2 \sqrt{\frac{\pi}{\mu}} \frac{e^{\mu t^2}}{(8\pi t)^{\frac{m}{2}}} e^{-||x-x'||^2/8t}, \quad (16)$$

and metastable domains of positive and negative Y are formed when the nonlinearity becomes important, at $g \simeq \sqrt{2\mu|\ln \epsilon|}$.

4. Conclusion

In systems with spatial degrees of freedom, the phenomenon of delayed bifurcation is accompanied by a noise-controlled formation of spatial structure. Here the stochastic PDE (3) was studied, whose realisations consist long sojourns near the zero configuration during which the shape of the spatial correlation function becomes Gaussian, followed by an abrupt change to the nonlinear régime, with the formation of domains of positive and negative Y . This spatial structure, formed near the critical value of the bifurcation parameter g when noise dominates, is frozen in at $g \simeq \sqrt{2\mu|\ln \epsilon|}$, when the characteristic length is $\sqrt{8t}$.

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